Student Response

This student understands that because the denominator of 5 is less, it means the parts are larger, and so the two parts are more of the whole (Principle 3).

How do you know that $\frac{2}{10}$ is less than $\frac{2}{5}$?

because 5 is smaller that way there is more for 2.

TEACHING TIP

Students should be helped to see that one of the main differences between fractions and counting numbers is the fact that there are always in-between fractions, but there are not always in-between counting numbers. You might ask students whether there are fractions between $\frac{2}{5}$ and $\frac{3}{5}$. Many will realize that $\frac{1}{2}$ works.

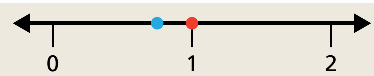
Principle 6: No matter what two different fractions are selected, there is a fraction in between.

Between $\frac{3}{5}$ and $\frac{4}{5}$ are an infinite number of fractions. You can rename $\frac{3}{5}$ and $\frac{4}{5}$ with greater denominators to find those in-between fractions, as shown below.

$\frac{3}{5} = \frac{6}{10}$ and $\frac{4}{5} = \frac{8}{10}$, so $\frac{7}{10}$ is between.

$\frac{3}{5} = \frac{12}{20}$ and $\frac{4}{5} = \frac{16}{20}$, so $\frac{13}{20}$, $\frac{14}{20}$, and $\frac{15}{20}$ are between, etc.

Cramer et al. (2017) have noted that students have difficulty ordering fractions on a number line when some fractions are less than 1 and some are greater than 1. Many students place $\frac{1}{2}$ at the red dot below, but $\frac{3}{4}$ at the blue dot. Notice they are seeing the whole as the full distance from 0 to 2 in one situation but not the other.



This suggests that it is important for teachers to ensure that they require students to order sets of fractions including both fractions less than 1 and greater than 1.

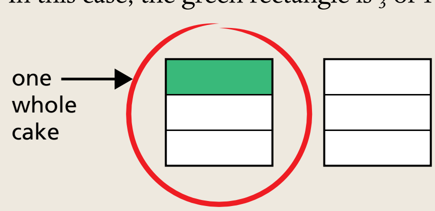
Relating Fractions to Decimals

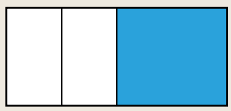
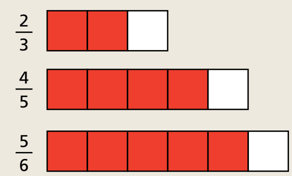
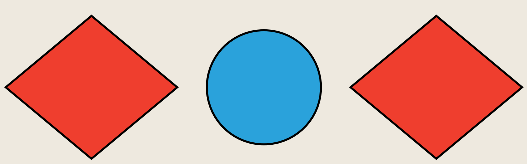
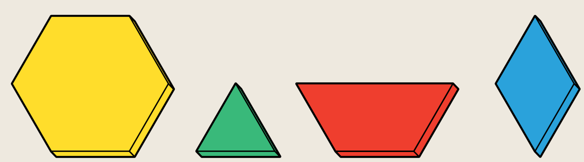
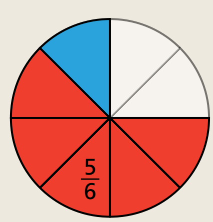
At some point, students will be ready to write a fraction as a decimal quantity. At the K–8 level, students begin by writing fractions like $\frac{3}{10}$ or $\frac{3}{100}$, with denominators as powers of ten, in decimal form. Then they write other fractions, with equivalents in this form, as decimals, for example, $\frac{1}{8} = \frac{125}{1000}$ as 0.125. But eventually, they learn to write any fraction as a decimal, using the division meaning for fractions. Since $\frac{2}{3}$ means $2 \div 3$, then the decimal for $\frac{2}{3}$ is calculated by dividing 2 by 3, usually using a calculator.

Common Errors and Misconceptions

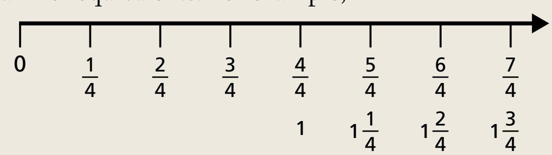
Many of the common errors in fraction work stem from a lack of understanding of the underlying principles. If students continue to make these kinds of errors and do not seem to understand what they are doing wrong, they should be using concrete and pictorial models of the fractions.

Fractions: Common Errors, Misconceptions, and Strategies

COMMON ERROR OR MISCONCEPTION	STRATEGY
Understanding the Whole in Fractions Students do not understand the importance of the whole in describing a fraction. For example, in the situation below, 2 cakes are each divided into thirds. Even though there are 6 shares, $\frac{1}{3}$ still describes $\frac{1}{3}$ of one cake. Many students see 6 parts and call each part $\frac{1}{6}$. 	Students should be encouraged to circle or highlight the whole unit and use language to reinforce the whole. In this case, the green rectangle is $\frac{1}{3}$ of 1 cake. 

COMMON ERROR OR MISCONCEPTION	STRATEGY
Equal Parts of a Region or Measure Students do not pay attention to the need for all parts to be equal when talking about parts of a length, area, volume, mass, or time. (Note that this is not the case when talking about parts of a set.) For example, the blue part of the rectangle below is not $\frac{1}{3}$ of the area even though it is 1 of 3 parts. 	Ensure that students have many opportunities to work with incorrectly divided wholes in context. For the example shown here, you might relate the fraction to a real context, such as a cake divided into 3 pieces. Ask if students think that the recipients will each be happy with their "thirds."
Basing Equivalence on the Number of Missing Parts Some students think that $\frac{2}{3}$, $\frac{4}{5}$, and $\frac{5}{6}$ are all equivalent since each is missing one part.	Remind students that we are talking about $\frac{2}{3}$, $\frac{4}{5}$, and $\frac{5}{6}$ of the same whole when comparing them. For this reason, using a region model might prove convincing. 
Parts of Sets Can Be Different Students are confused by sets containing different items. For example, students might not recognize the circle below as $\frac{1}{3}$ of the set of shapes.  The circle is $\frac{1}{3}$ of the set of shapes.	Many students do not have enough experience working with parts of sets. This can be remedied by providing more of these opportunities. Using pattern blocks for both fractions of a set and region can help; for example, the green block is $\frac{1}{6}$ of the yellow block (fraction of a region) but $\frac{1}{4}$ of the set of 4 blocks below (fraction of a set).  The green block is $\frac{1}{4}$ of this set of blocks.
Partitioning Circles in Difficult Situations It is relatively easy for students to partition circles into halves, fourths, and eighths, but not really easy to show other partitions. So to show sixths, some students might partition into 8 and just ignore 2 sections. 	Students should be encouraged to use models other than circles. As well, students might be reminded that they should ALWAYS identify what the whole is and make sure the whole is properly divided (whether on a number line or a shape or a capacity, etc.)
Increasing the Value of a Fraction Students have the misconception that a fraction always increases in value if the numerator and denominator are increased.	There are specific circumstances when this is true; for example, if both the numerator and denominator of a fraction less than 1 increase by the same amount, the value of the fraction increases. It is only by using a counter-example that students will see that adding (different) amounts to the numerator and denominator does not always increase the value of the fraction. For example, $\frac{5}{8} \longrightarrow \frac{5+1}{8+2} = \frac{6}{10} \quad \frac{6}{10} < \frac{5}{8}$ *(Continued)*

Fractions: Common Errors, Misconceptions, and Strategies (Continued)

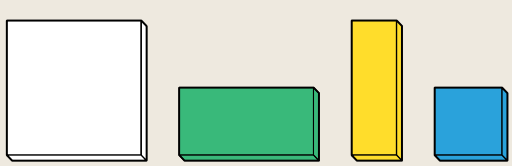
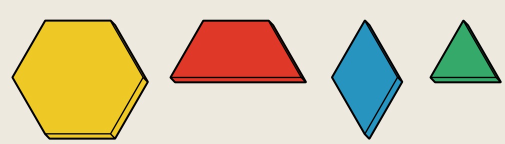
<table>
<tr><th>COMMON ERROR OR MISCONCEPTION</th><th>STRATEGY</th></tr>
<tr><td>Fractions Can Be Greater Than 1
Students have the misconception that all fractions are less than 1.</td><td>There are many real-world examples of fractions greater than 1, such as $1\frac{1}{2}$ hours and $2\frac{1}{2}$ dozen cookies. Students need many opportunities to work with mixed numbers and equivalent fractions in a variety of contexts.</td></tr>
<tr><td>Modeling Fractions Greater than 1 on Number Lines
Students have difficulty placing fractions on number lines when the fractions are greater than 1.</td><td>It might be helpful for students to count on a number line labeled with fractions only, and later add the whole number equivalents. For example,


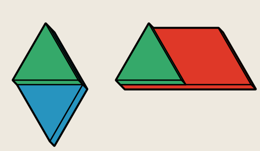
Later, students can think of a fraction greater than 1 as a mixed number, choose the segment in which to work, and then mentally translate what they would do in the segment from 0 to 1 to represent the fraction part of the mixed number.</td></tr>
</table>

Appropriate Manipulatives

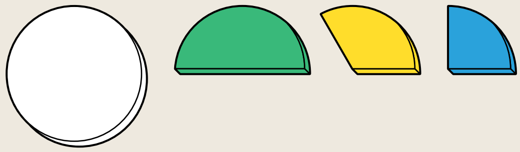
Many appropriate manipulative materials are available for work with fractions. It is essential that all students work with at least some of these materials and not rely on pictures. Even older students should be encouraged to use physical representations of fractions.

Fractions: Examples of Manipulatives

<table>
<tr><th>FRACTION PIECES</th><th>PATTERN BLOCKS</th></tr>
<tr><td>Fraction pieces are shapes, whether plastic, paper, or some other material, that are precut to show various fractions and are used to represent fractions of regions. Usually, the different fractions are different colors. The wholes upon which these fractions are based are not always the same shape. Often they are square, rectangular, or circular.

</td><td>Pattern blocks provide another model for representing fractions of a region. Pattern blocks work particularly well for showing halves, thirds, and sixths if the yellow hexagon block is considered 1 or the whole.


Note that the fraction representations can change, depending on which block is considered the whole.

The green block is $\frac{1}{2}$ of the blue block.
The green block is $\frac{1}{3}$ of the red block.</td></tr>
</table>

FRACTION PIECES



1 or whole $\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{4}$

PATTERN BLOCKS

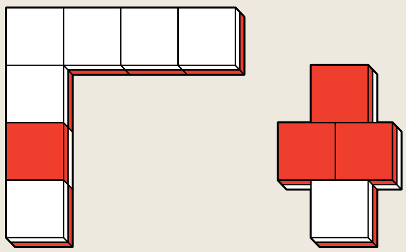
There is an alternate version of pattern blocks that makes it easier to show fourths.

In this case, a double hexagon is a whole.

1 whole $\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{4}$ $\frac{1}{6}$ $\frac{1}{12}$

SQUARE TILES

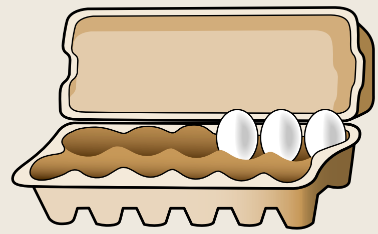
Square tiles come in different colors, and can be used to show many different fractions of a region. They can also be used to show fractions of a set.



$\frac{1}{7}$ red $\frac{3}{4}$ red

EGG CARTONS

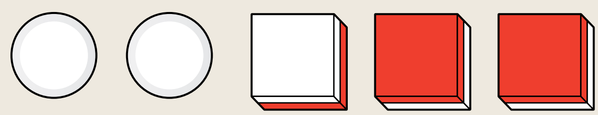
Egg cartons are useful for showing fractions with denominators that are factors of 12 (halves, thirds, fourths, sixths, and twelfths).



$\frac{3}{12}$, or $\frac{1}{4}$, of the carton is filled.

COUNTERS

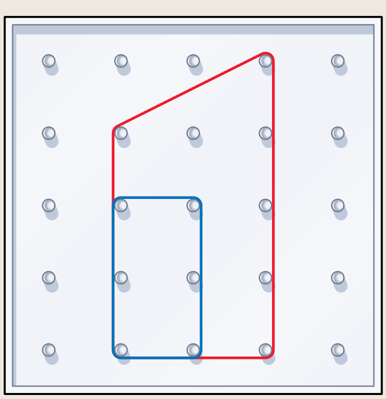
Counters in a variety of colors and shapes can be used to show fractions of sets. Two-sided counters with a different color on each side are also useful.



$\frac{3}{5}$ of the counters are white. $\frac{3}{5}$ of the counters are square.

GEOBOARDS AND SQUARE DOT OR GRID PAPER

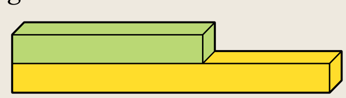
Geoboards allow students to represent fractions as parts of regions in a variety of ways. Square dot paper or grid paper can be used to record what is shown on the geoboard or as a pictorial equivalent. One of the advantages of the geoboard is that more unusual shapes and fractions, such as $\frac{2}{7}$, can be shown.

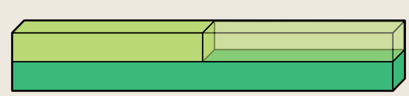


If the whole is 7 squares, 2 squares are $\frac{2}{7}$.

CUISENAIRE RODS

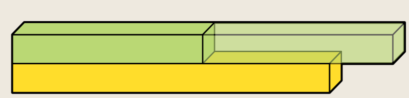
Cuisenaire rods are useful to show the concepts of fraction of a length and fraction as a ratio or comparison. As with pattern blocks, any rod can be considered the whole, so the fraction that each rod represents varies, depending on what is considered the whole, for example:





The light green rod is $\frac{3}{5}$ of the yellow rod but $\frac{1}{2}$ of the dark green rod.

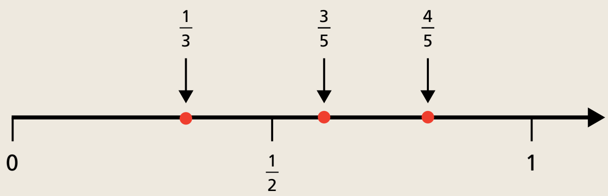
The rods can also be used to show fractions greater than 1 and mixed numbers, for example:



The yellow rod is $\frac{5}{3}$, or $1\frac{2}{3}$ of the light green rod.

NUMBER LINES

Number lines are useful for comparing and ordering fractions. Students can place fractions at appropriate spots on a number line using the benchmarks $0, \frac{1}{2}$, and 1.



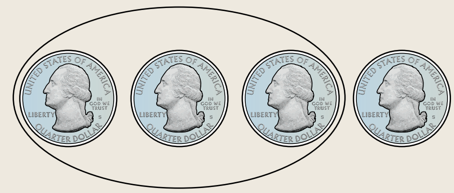
$\frac{4}{5}$ can be placed using its size relative to 0 $(\frac{0}{5})$ and 1 $(\frac{5}{5})$.

$\frac{1}{3}$ can be placed using its size relative to 0 and $\frac{1}{2}$.

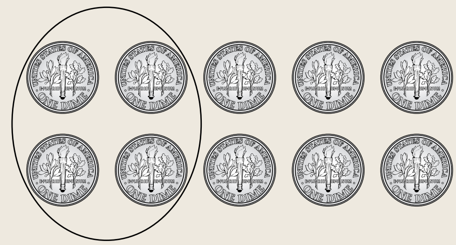
$\frac{3}{5}$ can be placed using its size relative to $\frac{1}{2}$ and 1.

MONEY

Money provides a value model that is very effective for students who have internalized money concepts.



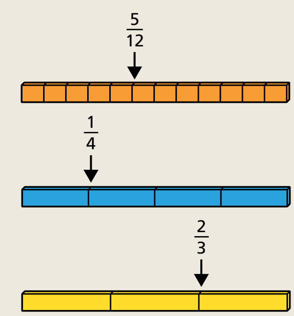
3 quarters is $\frac{3}{4}$ or $\frac{75}{100}$ of a dollar.



4 dimes is $\frac{4}{10}$, $\frac{40}{100}$, or $\frac{2}{5}$ of a dollar.

FRACTION BARS (OR STRIPS OR RECTANGLES)

Fraction bars are another length model for showing fractions of a measure. They can be made of plastic or paper and are sometimes called fraction strips or rectangles. The bars are all the same length, are predivided, and are sometimes color coordinated to show halves, thirds, fourths, fifths, sixths, eighths, tenths, and twelfths. These strips might be cut out of a fraction tower. If the tower is used on an interactive whiteboard, the fraction strips can be superimposed.



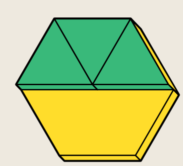
$\frac{5}{12}$ is greater than $\frac{1}{4}$ but less than $\frac{2}{3}$.

Modeling with Manipulatives

MODELING EQUIVALENCE

The equivalence of fractions can be shown using manipulatives. For example, using pattern blocks:

If the yellow block is 1 or the whole, each green block is $\frac{1}{6}$. So, 3 green blocks make up $\frac{3}{6}$, or $\frac{1}{2}$, of the yellow block.

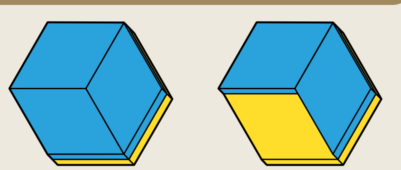


$\frac{3}{6} = \frac{1}{2}$

MODELING FRACTIONS GREATER THAN 1

Fractions greater than 1 can also be modeled using manipulatives. The equivalence of mixed numbers and their corresponding fractions also becomes apparent. For example, using pattern blocks:

If the yellow block is 1, then each blue block is $\frac{1}{3}$. So, 5 blue blocks are $\frac{5}{3}$, or $1\frac{2}{3}$, yellow blocks.



$\frac{5}{3} = 1\frac{2}{3}$